If diffusion from the α- to the β-phase is slow under even favourable circumstances, it follows that if the β-phase already contains gas the rate may become almost unmeasurable, for, as has been shown in this communication, a small change of concentration may produce a great change in the rate of solution.

There appears, therefore, every reason to suppose that the solution of hydrogen by palladium is a phenomenon quite analogous to the cases of sorption of gases by other solids, such as charcoal and celluloid, which have already been examined by various authors.

Conclusion.

The rate of solution of hydrogen by palladium is not a simple function of the concentration of gas in the metal. The rate curves consist of two portions (except in the case of palladium black), which have been interpreted as referring to solution in two different forms of the metal. The smooth rate curve for palladium black is taken to mean the almost complete absence of one of these forms.

Note on a Functional Equation employed by Sir George Stokes.

By Sir James Stirling, F.R.S.

(Received February 24,—Read March 12, 1914.)

1. In a paper by Sir George Stokes "On the Intensity of the Light Reflected from or Transmitted through a Pile of Plates,"* the author forms and solves two functional equations, viz.:—

$$\phi(m+n) = \phi(m) + \frac{\phi(n)\,\{\psi(m)\}^2}{1-\phi(m)\,\phi(n)}, \qquad (1)$$

$$\psi(m+n) = \frac{\psi(m)\,\psi(n)}{1-\phi(m)\,\phi(n)}. \qquad (2)$$

The symbols m, n represent, as was required by the object of the paper, positive integers; but in the course of the investigation he deals with them as continuous variables, solves the equations (1) and (2) on that footing, and finally verifies the equations when m, n are integers. The two equations are treated as independent, and the results are thus expressed,

$$\frac{\phi(m)}{\sin m\beta} = \frac{\psi(m)}{\sin \alpha} = \frac{1}{\sin(\alpha+m\beta)}, \qquad (3)$$

* Published in 'Roy. Soc. Proc.' for 1862 ; see his 'Math. and Phys. Papers,' vol. 4, pp. 145, 148, 149.

where α, β are constants.   In what follows m, n will be taken to be ordinary continuous variables.

2. The main object of the paper of Sir George Stokes is physical, not mathematical; and the purpose of the present note is to call attention to some mathematical points not explicitly dealt with by him, viz. :—

(*a*) The two functional equations are not independent, equation (2) being capable of being deduced from equation (1);

(*b*) The results (3) may be arrived at from equation (1) *alone*, and constitute the general solution of that equation.   No such general solution of equation (2) has been obtained.

3. If equation (1) be differentiated with respect to n, the result, after some obvious reductions, is

$$\phi'(m+n) = \phi'(n) \{\psi(m)\}^2 / \{1 - \phi(m)\phi(n)\}^2. \tag{4}$$

If m and n be interchanged in the last equation, the result is

$$\phi'(m+n) = \phi'(m) \{\psi(n)\}^2 / \{1 - \phi(m)\phi(n)\}^2, \tag{5}$$

so that $\{\psi(m)\}^2 / \phi'(m) = \{\psi(n)\}^2 / \phi'(n) = A$, a constant,

or $$\{\psi(m)\}^2 = A\phi'(m). \tag{6}$$

Hence $\{\psi(m+n)\}^2 = A\phi'(m+n),$

[by (4)] $= A\phi'(n) \{\psi(m)\}^2 / \{1 - \phi(m)\phi(n)\}^2$

[by (6)] $= \{\psi(n)\}^2 \{\psi(m)\}^2 / \{1 - \phi(m)\phi(n)\}^2,$

and therefore $$\psi(m+n) = \pm \frac{\psi(m)\psi(n)}{1 - \phi(m)\phi(n)},$$

which is equation (2), with an ambiguity of sign.   This ambiguity also arises in the course of Sir George Stokes' investigation and is dealt with by him.

It follows that every value of $\phi(m)$, $\psi(m)$, which satisfies equation (1) satisfies also equation (2); but it is not necessarily true that every value of $\phi(m)$, $\psi(m)$, which satisfies equation (2) also satisfies equation (1).

4. If now in equation (1), and that derived from it by interchanging m, n, there be substituted for $\{\psi(m)\}^2$, $\{\psi(n)\}^2$, their values $A\phi'(m)$, $A\phi'(n)$, it appears that

$\phi(m) + \phi(n) A\phi'(m) / \{1 - \phi(m)\phi(n)\}$

$$= \phi(n) + \phi(m) A\phi'(n) / \{1 - \phi(m)\phi(n)\},$$

which, when multiplied on both sides by $1 - \phi(m)\phi(n)$, becomes

$$\phi(m)\{1 - \phi(m)\phi(n)\} + A\phi'(m)\phi(n) = \phi(n)\{1 - \phi(m)\phi(n)\} + A\phi(m)\phi'(n),$$

or

$$\phi(m)\{1 - A\phi'(n)\} - \phi(n)\{1 - A\phi'(m)\} - \phi(m)\phi(n)\{\phi(m) - \phi(n)\} = 0;$$

or, finally, $\qquad \dfrac{1-\mathrm{A}\,\phi'(n)}{\phi(n)}+\phi(n)-\left\{\dfrac{1-\mathrm{A}\,\phi'(m)}{\phi(m)}+\phi(m)\right\}=0,$

whence $\qquad\qquad \dfrac{1-\mathrm{A}\,\phi'(m)}{\phi(m)}+\phi(m)=\text{a constant},$

which (after Sir G. Stokes) may be written $2\cos\alpha$, and consequently

$$\phi'(m)=\mathrm{A}^{-1}\left[\{\phi(m)\}^2-2\cos\alpha\,\phi(m)+1\right], \qquad (7)$$

a differential equation identical in form with that ultimately arrived at by Stokes, his constant β standing for $\mathrm{A}^{-1}\sin\alpha$.* The results (3) are obtained by integrating this equation. A fresh constant is thus introduced, and its value is to be found by ascertaining the value of $\phi(0)$. If in equation (1) the value 0 is assigned to n, it appears that

$$0=\phi(0)\,\{\psi(m)\}^2/\{1-\phi(m)\,\phi(0)\},$$

whatever m may be ; and consequently, *in general*, the value of $\phi(0)$ is zero.

The results (3) are thus deduced from equation (1) alone, without recourse to equation (2).

5. The solution of the two functional equations was proposed by Sir George Stokes in a Smith's Prize Examination Paper for 1860,† and was introduced as an example into Boole's ' Finite Differences '; but the values of $\phi(x)$ and $\psi(x)$ given in that work are deficient in generality. This defect, it is right to say, is not attributable to the late Professor Boole.

---

* See ' Math. and Phys. Papers,' vol. 4, p. 149, equation (10).
† See ' Math. and Phys. Papers,' vol. 5, p. 334, Question 14.